CURVATURE INVARIANTS IN ALGEBRAICALLY SPECIAL SPACETIMES

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Let us define a curvature invariant of the order k as a scalar polynomial constructed from $g_{\alpha\beta}$, the Riemann tensor $R_{\alpha\beta\gamma\delta}$, and covariant derivatives of the Riemann tensor up to the order k. According to this definition, the Ricci curvature scalar R or the Kretschmann curvature scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ are curvature invariants of the order zero and $R_{\alpha\beta\gamma\delta;\varepsilon}R^{\alpha\beta\gamma\delta;\varepsilon}$ is a curvature invariant of the order 1.

We consider only vacuum spacetimes so that the Riemann tensor is equal to the Weyl tensor. An arbitrary curvature invariant can thus be expressed in terms of the Weyl spinor Ψ_{ABCD} and its covariant derivatives.

We can use a standard basis o_A, ι_A, which satisfies

$$o_A\iota^A = 1, \quad o_A o^A = 0, \quad \iota_A\iota^A = 0, \tag{1}$$

to decompose the Weyl spinor in the form

$$\begin{aligned}\Psi_{ABCD} &= \Psi_0\iota_A\iota_B\iota_C\iota_D - 4\Psi_1 o_{(A}\iota_B\iota_C\iota_{D)} + 6\Psi_2 o_{(A}o_B\iota_C\iota_{D)} \\ &\quad -4\Psi_3 o_{(A}o_B o_C\iota_{D)} + \Psi_4 o_A o_B o_C o_D \ .\end{aligned} \tag{2}$$

We concentrate on spacetimes of the Petrov type-*III* and *N*. Since three principal spinors of Ψ_{ABCD} coincide in type-*III* spacetimes, it is convenient to choose this repeated principal spinor as the basis spinor o_A. Then

$$\Psi_{ABCD} = o_{(A}o_B o_C\delta_{D)} \ , \tag{3}$$

and

$$\Psi_0 = \Psi_1 = \Psi_2 = 0 \ . \tag{4}$$

The Weyl spinor has the form

$$\Psi_{ABCD} = -4\Psi_3 o_{(A}o_B o_C\iota_{D)} + \Psi_4 o_A o_B o_C o_D \ . \tag{5}$$

In type-*N* spacetimes, all four principal spinors of Ψ_{ABCD} coincide and thus also Ψ_3 vanish.

Now it is obvious that all curvature invariants of the order zero vanish for type-*III* and type-*N* vacuum spacetimes as a consequence of the relation $o_A o^A = 0$. The question arises, whether there exist some non-vanishing curvature invariants of higher orders. Let us summarize main results - corresponding proofs and calculations can be found in the papers[1,2] for type-*N* and type-*III* spacetimes, respectively.

Table 1: Curvature invariants in vacuum spacetimes (0 - vanish, 1 - do not vanish)

Petrov type	I, II, D	III	III	N	N
expansion and twist		$\rho \neq 0$	$\rho = 0$	$\rho \neq 0$	$\rho = 0$
curvature invariants of order 0	1	0	0	0	0
curvature invariants of order 1	1	1	0	0	0
curvature invariants of order 2	1	1	0	1	0
curvature invariants of order > 2	1	1	0	1	0

It turns out that the results depend on whether the Newman-Penrose coefficient $\rho = \theta + i\omega$ vanishes or not (i.e., whether the corresponding spacetime has a non-vanishing expansion θ or twist ω).

For $\rho = 0$ the following can be shown:

Proposition

In type-N or type-III vacuum spacetimes, admitting a non-expanding and non-twisting null geodesic congruence, all curvature invariants constructed from the Riemann tensor and its covariant derivatives of arbitrary order vanish.

In the case with $\rho \neq 0$, all curvature invariants of the first order in type-N vacuum spacetimes vanish but there exists a non-vanishing curvature invariant of the first order in type-III vacuum spacetimes, which, in appropriately chosen spinor basis, can be expressed as follows:

$$R^{\alpha\beta\gamma\delta;\varepsilon}R_{\alpha\mu\gamma\nu;\varepsilon}R^{\lambda\mu\rho\nu;\sigma}R_{\lambda\beta\rho\delta;\sigma} = (48\rho\bar{\rho}\Psi_3\bar{\Psi}_3)^2 \ . \tag{6}$$

In the type-N vacuum spacetimes with expansion or twist (i.e., $\rho \neq 0$), the first non-vanishing curvature invariant is of the second order;

$$R^{\alpha\beta\gamma\delta;\varepsilon\phi}R_{\alpha\mu\gamma\nu;\varepsilon\phi}R^{\lambda\mu\rho\nu;\sigma\tau}R_{\lambda\beta\rho\delta;\sigma\tau} = (48\rho^2\bar{\rho}^2\Psi_4\bar{\Psi}_4)^2. \tag{7}$$

Properties of curvature invariants for different Petrov types are summarized in Table 1.

The non-vanishing invariant (7) can be used to show that twisting and diverging type-N solutions analyzed recently[3] cannot represent smooth radiation fields outside bounded sources. Solutions with invariants of all orders vanishing may be of importance from quantum perspective[4].

References

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